Unifying the notions of an element prime to another element and an element primary to another element under one frame in multiplicative lattices

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Dedicated to the memory of Professor C. S. Manjarekar

Abstract: In this paper, we introduce the notion of an element δ-primary to another element in a multiplicative lattice L. We obtain its various characterizations. We investigate many of its properties.

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1 Introduction and Preliminaries

Ward M. and Dilworth R. P. in [7] (also see [3] by Dilworth R. P.) introduced the concept of multiplicative lattices to provide an abstract framework for studying the ideal theory of commutative rings with identity. In the recent years, the study of multiplicative lattices has attracted significant attention from many researchers.

Manjarekar C. S. and Bingi A. V. in [4] introduced the notion of a δ-primary element in a multiplicative lattice L to unify the notions of prime element and primary element in L, under one frame where δ is an expansion function on L. In this paper, we work on similar lines.

The notion of an element prime to another element in a multiplicative lattice L is introduced by Alarcon F. et. al. in [1]. Further, the notion of an element primary to another element in a multiplicative lattice L is introduced by Manjarekar C. S. and Chavan N. S. in [5]. The main aim of this paper is to unify the notions of an element prime to another element and an element primary to another element in a multiplicative lattice L, under one frame by introducing the concept of "an element δ-primary to another element" in a multiplicative lattice L and then investigate its properties. This concept introduced has potential for future study and the results obtained, though elementary, are significant. This paper is a good source for beginners to conduct research work on multiplicative lattices. Additionally, this paper aims to stimulate emerging researchers to explore the field of multiplicative lattices.

Let us recall a few definitions, notations and terminologies in multiplicative lattices that will be needed in this paper. A multiplicative lattice is a complete lattice $(L, \leqslant, 0, 1)$ provided with commutative, associative and join distributive multiplication (denoted by $\cdot$) in which the largest element 1 acts as a multiplicative identity, where $\leqslant$ is a partial order on L. The product of any two elements a and b of L will be denoted by ab instead of $a \cdot b$ for convenience. $\mathbb{Z}_+$ denotes the set of all natural numbers. For $A \subseteq L$, $\vee A$ = the join of A denotes the supremum of A and $\wedge A$ = the meet of A denotes the infimum of A. For brevity, we shall write $a \vee b$ for $\vee\{a, b\}$ and $a \wedge b$ for $\wedge\{a, b\}$ where $a, b \in L$. Note that a^n denotes $a \cdot ... \cdot a$ (repeated n times) where $a \in L$ and $n \in \mathbb{Z}_+$. For $a, b \in L$, $(a : b) = \vee\{x \in L \mid xb \leqslant a\}$ where $(a : b)$ is read as the residuation of a by b. An element $a \in L$ is called compact if for $X \subseteq L$, $a \leqslant \vee X$ imply the existence of a finite number of elements $a_1, \cdots, a_n$ in X such that $a \leqslant a_1 \vee \cdots \vee a_n$. If each element of L is a join of compact elements of L, then L is called a compactly generated lattice or simply a CG-lattice. L_* denotes the set of all compact elements of L. An element $a \in L$ is said to be proper if $a < 1$. The radical of $a \in L$ is denoted by $\sqrt{a}$ and is defined as $\vee\{x \in L_* \mid x^n \leqslant a, \text{for some } n \in \mathbb{Z}_+\}$. According to [6], $a \leqslant \sqrt{a}\ \forall\ a \in L$ and $a \leqslant b$ implies $\sqrt{a} \leqslant \sqrt{b}\ \forall\ a, b \in L$. A proper element $p \in L$ is called a prime element if $ab \leqslant p$ imply either $a \leqslant p$ or $b \leqslant p$ where $a, b \in L$. A proper element $p \in L$ is called a primary element if $ab \leqslant p$ imply either $a \leqslant p$ or $b \leqslant \sqrt{p}$ where $a, b \in L_*$. In a lattice L, a family $\{a_i\} \subseteq L$ is an ascending (or descending) chain, if $a_1 \leqslant a_2 \leqslant a_3 \leqslant \cdots$ (or $a_1 \geqslant a_2 \geqslant a_3 \geqslant \cdots$). A lattice isomorphism is a bijective function between two lattices that preserves the order structure. Throughout this paper, L denotes a compactly generated multiplicative lattice with 1 compact in which every finite product of compact elements is compact. For general background and terminology in multiplicative lattices, one can refer [1] and [5].

The content of the paper is as follows: In Section 2, we introduce the notion of an element δ-primary to another element in a multiplicative lattice L (see Definition 2.4) and obtain its characterizations (see Theorems 2.5, 2.6, 2.12, 2.22). If an element $b \in L$ is δ_0-primary to a proper element $p \in L$, then b is δ_1-primary to p. But its converse need not be true as illustrated in Example 2.7. We define the notion of an element q-primary to another element in L (see Definition 2.10). We investigate under which condition(s)

① an element is δ-primary to another element in L (see Theorem 2.14).

② equality holds in $\delta(p) \leqslant \delta(p : a)$ (see Theorem 2.15).

③ an element is δ-primary to $(p : q) \in L$ (see Theorem 2.16).

④ an element is δ-primary to the join of proper elements in L (see Theorem 2.17).

⑤ an element is δ-primary to the meet of proper elements in L (see Theorem 2.19).

We conclude this paper with results related to global property of an expansion function δ on L.

2 An Element δ-primary To Another Element In L

Initially, we recall the notion of an expansion function δ on a multiplicative lattice L which assigns each element $a \in L$ to a unique element of L.

Definition 2.1. *(Manjarekar C. S. and Bingi A. V. [4]). An expansion function on a multiplicative lattice L is a function $\delta : L \longrightarrow L$ which satisfies the following two conditions:*
①. $a \leqslant \delta(a)$ *for all* $a \in L$.
②. $a \leqslant b$ *imply* $\delta(a) \leqslant \delta(b)$ *for all* $a,\ b \in L$.

The reader can verify that the following functions on L are expansion functions.

1. the identity function $\delta_0 : L \longrightarrow L$ defined as $\delta_0(a) = a$ for all $a \in L$.
2. the function $\delta_1 : L \longrightarrow L$ defined as $\delta_1(a) = \sqrt{a}$ for all $a \in L$.

Now, we recall the definitions related to the notions of an element prime to another element and an element primary to another element in a multiplicative lattice L.

Definition 2.2. *(Alarcon F. et. al. [1]) An element $b \in L$ is said to be prime to a proper element $p \in L$ if for all $x \in L$, $xb \leqslant p$ imply $x \leqslant p$.*

Definition 2.3. *(Manjarekar C. S. and Chavan N. S. [5]) An element $b \in L$ is said to be primary to a proper element $p \in L$ if for all $x \in L$, $xb \leqslant p$ imply $x \leqslant \sqrt{p}$.*

Now, we unify the notions of an element prime to another element and an element primary to another element in a multiplicative lattice L, under one frame by defining the notion of an element δ-primary to another element in L.

Definition 2.4. *Given an expansion function δ on L, an element $b \in L$ is said to be δ-primary to a proper element $p \in L$ if for all $x \in L$, $xb \leqslant p$ imply $x \leqslant \delta(p)$.*

The following theorem characterizes the notion of an element δ_0-primary to another element and relate it with the notion of an element prime to another element in L.

Theorem 2.5. *An element $b \in L$ is δ_0-primary to a proper element $p \in L$ if and only if b is prime to p.*

Proof. The proof is obvious. □

The following theorem characterizes the notion of an element δ_1-primary to another element and relate it with the notion of an element primary to another element in L.

Theorem 2.6. *An element $b \in L$ is δ_1-primary to a proper element $p \in L$ if and only if b is primary to p.*

Clearly, if an element $b \in L$ is δ_0-primary to a proper element $p \in L$, then b is δ_1-primary to p. But the converse is not true as shown in the following example.

Example 2.7. *Consider the lattice L of ideals of the ring $R = < Z_{12}\ ,\ +_{12}\ ,\ \times_{12} >$. Then the only ideals of R are the principal ideals (0),(1),(2),(3),(4),(6). Clearly, $L = \{(0), (1), (2), (3), (4), (6)\}$ is a compactly generated multiplicative lattice. It is easy to see that the element $(6) \in L$ is δ_1-primary to a proper element $(4) \in L$ but (6) is not δ_0-primary to (4) .*

Definition 2.8. *An element $b \in L$ is said to be semiprimary to a proper element $p \in L$ if b is prime to $\sqrt{p}$.*

It is easy to see that if an element $b \in L$ is prime to a proper element $p \in L$, then b is semiprimary to p.

Theorem 2.9. *If an element $b \in L$ is semiprimary to a proper element $p \in L$, then b is δ_0-primary to $\sqrt{p}$.*

Proof. The proof is obvious. □

Definition 2.10. *If an element $b \in L$ is primary to a proper element $p \in L$ and b is prime to $q = \sqrt{p}$, then b is said to be q-primary to p.*

Theorem 2.11. *If an element $b \in L$ is q-primary to a proper element $p \in L$, then b is δ_1-primary to p and b is δ_0-primary to q.*

Proof. The proof is obvious. □

Now we obtain a characterization of an element δ-primary to another element in L.

Theorem 2.12. *Let p be a proper element of L and $b \in L$. Then the following statements are equivalent:*

(1) b is δ-primary to p.

(2) $(p : b) \leqslant \delta(p)$.

(3) for all $r \in L_*$ $rb \leqslant p$ imply $r \leqslant \delta(p)$.

Proof. (1) $\Longrightarrow$ (2). Suppose (1) holds. Let $h \in L_*$ be such that $h \leqslant (p : b)$. Then $hb \leqslant p$. As b is δ-primary to p, we have $h \leqslant \delta(p)$. By Lemma 2.3.13 from [2], we have $(p : b) \leqslant \delta(p)$.

(2) $\Longrightarrow$ (3). Suppose (2) holds. Let $rb \leqslant p$ where $r \in L_*$. Then $r \leqslant (p : b)$ and hence by (2), we have $r \leqslant \delta(p)$.

(3) $\Longrightarrow$ (1). Suppose (3) holds. Let $xb \leqslant p$ where $x \in L$. Then as L is compactly generated, there exist $y' \in L_*$ such that $y' \leqslant x$. Let $y \in L_*$ be such that $y \leqslant x$. Then $(y \vee y') \in L_*$ such that $(y \vee y')b \leqslant p$. So by (3), we have $(y \vee y') \leqslant \delta(p)$ which implies $x \leqslant \delta(p)$ and therefore b is δ-primary to p. □

Definition 2.13. *Given any two expansion functions $\gamma,\ \delta : L \longrightarrow L$, we define, $\delta \leqslant \gamma$ if $\delta(a) \leqslant \gamma(a)$ for each $a \in L$.*

The next result shows that if an element $b \in L$ is prime to a proper element $p \in L$, then b is δ-primary to p, for any expansion function δ on L.

Theorem 2.14. *Let δ and γ be expansion functions on L such that $\delta \leqslant \gamma$. If an element $b \in L$ is δ-primary to a proper element $p \in L$, then b is γ-primary to p. In particular, if an element $b \in L$ is prime to a proper element $p \in L$, then b is δ-primary to p, for every expansion function δ on L.*

Proof. Assume that an element $b \in L$ is δ-primary to a proper element $p \in L$. Let $xb \leqslant p$ for $x \in L$. Then $x \leqslant \delta(p) \leqslant \gamma(p)$ and so b is γ-primary to p. Next, let an element $b \in L$ be prime to a proper element $p \in L$. Then by Theorem 2.5, b is δ_0-primary to p. For any expansion function δ on L, we have $p \leqslant \delta(p)$ and so $\delta_0(p) \leqslant \delta(p)$. Thus b is δ-primary to p, for every expansion function δ on L. □

It is easy to see that $\delta(p) \leqslant \delta(p : a)$ for all $a \in L$. Under which condition equality holds, is given in the following result.

Theorem 2.15. *Let δ be an expansion function on L. If an element $b \in L$ is prime to a proper element $p \in L$, then $\delta(p) = \delta(p : a)$ for all $a \in L$ such that $b \leqslant a$. Further, $(p : a) \leqslant \delta(p)$.*

Proof. Let $b \leqslant a$ in L. Then $(p : a) \leqslant (p : b)$ and hence $(p : a)b \leqslant p$. It follows that $(p : a) \leqslant p$, as $b \in L$ is prime to a proper element $p \in L$. So $\delta(p : a) \leqslant \delta(p)$. But it is easy to see that $\delta(p) \leqslant \delta(p : a)$ and hence $\delta(p) = \delta(p : a)$. Now as $pa \leqslant p$, we have $p \leqslant (p : a)$. But $(p : a) \leqslant p$ and hence $(p : a) = p$. Thus $(p : a) \leqslant \delta(p)$ as $p \leqslant \delta(p)$. □

The reader can relate Theorem 2.15 with Theorem 2.14 and Theorem 2.12.

The following theorem tells us when an element $b \in L$ is δ-primary to an element $(p : q) \in L$.

Theorem 2.16. *Let δ be an expansion function on L. If an element $b \in L$ is prime to a proper element $p \in L$ and an element $q \in L$ is δ-primary to p, then b is δ-primary to $(p : q)$.*

Proof. Let $xb \leqslant (p : q)$ for $x \in L$. Then as b is prime to p and $(xq)b \leqslant p$, we have $xq \leqslant p$. Further, as q is δ-primary to p and $xq \leqslant p$, we have $x \leqslant \delta(p)$. But $p \leqslant (p : q)$ and hence $\delta(p) \leqslant \delta(p : q)$. It follows that $x \leqslant \delta(p : q)$ and we are done. □

In the next two results, we investigate for conditions needed so that an element is δ-primary to the join and meet of proper elements in L.

Theorem 2.17. *Let $\{p_i \mid i \in \triangle\}$ be a chain of proper elements in L such that an element $b \in L$ is δ-primary to each p_i $(i \in \triangle)$ where δ is an expansion function on L. Then b is δ-primary to $p = \bigvee_{i \in \triangle} p_i$.*

Proof. Since $1 \in L$ is compact, $\bigvee_{i \in \triangle} p_i = p \neq 1$. Let $xb \leqslant p$ for all $x \in L$. Then as $\{p_i \mid i \in \triangle\}$ is a chain, we have $xb \leqslant p_i$ for some $i \in \triangle$. As an element $b \in L$ is δ-primary to each proper element p_i $(i \in \triangle)$, it follows that $x \leqslant \delta(p_i)$. Since $p_i \leqslant p$, we have $\delta(p_i) \leqslant \delta(p)$ and so $x \leqslant \delta(p)$. Hence b is δ-primary to $p = \bigvee_{i \in \triangle} p_i$. □

Now, we recall the definition of meet preserving property of an expansion function δ on L.

Definition 2.18. *(Manjarekar C. S. and Bingi A. V. [4]). An expansion function δ on a multiplicative lattice L is said to have meet preserving property if for all $a,\ b \in L$, $\delta(a \wedge b) = \delta(a) \wedge \delta(b)$.*

Theorem 2.19. *Let the expansion function δ on L have the meet preserving property. If an element $b \in L$ is δ-primary to the proper elements $p_1, \cdots, p_n$ of L then b is δ-primary to $p = \bigwedge_{i=1}^{n} p_i$.*

Proof. As the expansion function δ on L has the meet preserving property, it is easy to see that $\delta(p) = \delta(\bigwedge_{i=1}^{n} p_i) = \bigwedge_{i=1}^{n} \delta(p_i)$. Clearly, $\bigwedge_{i=1}^{n} p_i = p \neq 1$. Let $xb \leqslant p$ for $x \in L$. Then $xb \leqslant p_i$ for all $i = 1, \cdots, n$. As b is δ-primary to each p_i $(i = 1, \cdots, n)$, we have $x \leqslant \delta(p_i)$ for all $i = 1, \cdots, n$ and so $x \leqslant \bigwedge_{i=1}^{n} \delta(p_i) = \delta(p)$. Hence b is δ-primary to $p = \bigwedge_{i=1}^{n} p_i$. □

Next, we recall the definition of a global property of an expansion function on multiplicative lattices.

Definition 2.20. *(Manjarekar C. S. and Bingi A. V. [4]). Let L_1 and L_2 be multiplicative lattices. An expansion function δ on L_1 and L_2 is said to have global property if for any lattice isomorphism $f : L_1 \longrightarrow L_2$, $\delta(f^{-1}(a)) = f^{-1}(\delta(a))$ for all $a \in L_2$.*

Clearly, the expansion function δ_0 has the global property. We conclude this paper with the following result whose consequence gives one more characterization of an element δ-primary to another element in L.

Lemma 2.21. *Let the expansion function δ on L_1 and L_2 have the global property and $f : L_1 \longrightarrow L_2$ be a lattice isomorphism where L_1 and L_2 are multiplicative lattices. If an element $b \in L_2$ is δ-primary to a proper element of $p \in L_2$, then $f^{-1}(b) \in L_1$ is δ-primary to $f^{-1}(p) \in L_1$.*

Proof. Let $y \cdot (f^{-1}(b)) \leqslant f^{-1}(p)$ for $y \in L_1$. Then $f(y) \cdot b \leqslant p$. As b is δ-primary to p in L_2, we have $f(y) \leqslant \delta(p)$. Now the global property of δ gives $y \leqslant f^{-1}(\delta(p)) = \delta(f^{-1}(p))$ showing that $f^{-1}(b)$ is δ-primary to $f^{-1}(p)$ in L_1. □

Theorem 2.22. *Let the expansion function δ on L_1 and L_2 have the global property and $f : L_1 \longrightarrow L_2$ be a lattice isomorphism where L_1 and L_2 are multiplicative lattices. Then an element $b \in L_1$ is δ-primary to a proper element $p \in L_1$ if and only if $f(b) \in L_2$ is δ-primary to $f(p) \in L_2$.*

Proof. Assume that an element $b \in L_1$ is δ-primary to a proper element $p \in L_1$. Clearly, the global property of δ gives $\delta(p) = \delta(f^{-1}(f(p))) = f^{-1}(\delta(f(p)))$. Then since f is onto, we have $f(\delta(p)) = \delta(f(p))$. Now, let $y \cdot f(b) \leqslant f(p)$ for $y \in L_2$. Then there exist $x \in L_1$ such that $f(x) = y$. So $f(xb) = f(x) \cdot f(b) = y \cdot f(b) \leqslant f(p)$. As b is δ-primary to p in L_1 and $xb \leqslant p$, we have $x \leqslant \delta(p)$. So $y = f(x) \leqslant f(\delta(p)) = \delta(f(p))$ showing that $f(b)$ is δ-primary to $f(p)$ in L_2. The converse follows from Lemma 2.21. □

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